Math 217 Fall 2025

Quiz 24 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_r\} \subset \mathbb{R}^n$ is *orthonormal* provided that ...

   > **Solution:** Each vector has unit length and distinct vectors are orthogonal, i.e.
   >
   > $$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

   (b) The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is ...

   > **Solution:** Its Euclidean norm:
   >
   > $$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2} \quad \text{for } \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

   (c) Suppose V and W are vector spaces and $T : V \to W$ is linear. The *image* of T is ...

   > **Solution:** The set of all outputs:
   >
   > $$\operatorname{im}(T) = \{T(v) \in W : v \in V\},$$
   >
   > which is a subspace of W.

   (d) Suppose V and W are vector spaces and $T : V \to W$ is linear. The *kernel* of T is ...

   > **Solution:** The set of vectors mapped to $\vec{0}$:
   >
   > $$\ker(T) = \{v \in V : T(v) = \vec{0}\},$$
   >
   > which is a subspace of V.

2. (a) Suppose V is a vector space and $v_0 \in V$. Suppose also that $v_1, \ldots, v_m$ are linearly independent vectors in V. Show that

$$v_0 \in \operatorname{Span}(v_1, \ldots, v_m) \iff \{v_0, v_1, \ldots, v_m\} \text{ is linearly dependent.}$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: ($\Rightarrow$) If $v_0 = \sum_{i=1}^m c_i v_i$, then

$$v_0 - \sum_{i=1}^m c_i v_i = 0$$

is a nontrivial linear relation among $v_0, v_1, \ldots, v_m$, hence they are linearly dependent. ($\Leftarrow$) If $a_0 v_0 + \sum_{i=1}^m a_i v_i = 0$ with not all a_i zero and if $a_0 = 0$, then $\sum_{i=1}^m a_i v_i = 0$ is a nontrivial relation among $v_1, \ldots, v_m$, contradicting linear independence. Thus $a_0 \neq 0$ and

$$v_0 = -\sum_{i=1}^m \frac{a_i}{a_0} v_i \in \operatorname{Span}(v_1, \ldots, v_m).$$

(b) Suppose V is n-dimensional with bases $\mathcal{A}$ and $\mathfrak{B}$. For $v \in V$, let $L_{\mathfrak{B}}(v) = [v]_{\mathfrak{B}} \in \mathbb{R}^n$ and $L_{\mathcal{A}}(v) = [v]_{\mathcal{A}} \in \mathbb{R}^n$. Show that the k^{th} column of the change-of-basis matrix $S_{\mathfrak{B}\to\mathcal{A}}$ is

$$(L_{\mathcal{A}} \circ L_{\mathfrak{B}}^{-1})(e_k).$$

Solution: By definition, the change-of-basis matrix $S_{\mathfrak{B}\to\mathcal{A}}$ satisfies

$$[v]_{\mathcal{A}} \;=\; S_{\mathfrak{B}\to\mathcal{A}}\,[v]_{\mathfrak{B}} \qquad \text{for all } v \in V.$$

Apply this to the $\mathfrak{B}$-basis vector b_k, for which $[b_k]_{\mathfrak{B}} = e_k$:

$$S_{\mathfrak{B}\to\mathcal{A}}\,e_k \;=\; [b_k]_{\mathcal{A}}.$$

Since $b_k = L_{\mathfrak{B}}^{-1}(e_k)$, we get

$$\boxed{(k\text{-th column of } S_{\mathfrak{B}\to\mathcal{A}}) \;=\; S_{\mathfrak{B}\to\mathcal{A}} e_k \;=\; L_{\mathcal{A}}\big(L_{\mathfrak{B}}^{-1}(e_k)\big) \;=\; (L_{\mathcal{A}} \circ L_{\mathfrak{B}}^{-1})(e_k)}.$$

Equivalently,

$$S_{\mathfrak{B}\to\mathcal{A}} = \big[\,[b_1]_{\mathcal{A}} \;\cdots\; [b_n]_{\mathcal{A}}\,\big].$$

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Consider the two bases for a subspace $V \subset \mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 11 \\ 12 \end{bmatrix} \right\}, \qquad \mathcal{A} = \left\{ \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/13 \\ 3/13 \\ 12/13 \end{bmatrix} \right\}.$$

The orthonormal basis $\mathcal{A}$ may be obtained from $\mathcal{B}$ by Gram–Schmidt.

Solution: **TRUE.** First vector: $\vec{a}_1 = \frac{1}{\|\vec{b}_1\|}\vec{b}_1 = \frac{1}{5}(3, 4, 0)^T = (3/5, 4/5, 0)^T$.

Second step: $\vec{b}_2' = \vec{b}_2 - \operatorname{proj}_{\vec{b}_1} \vec{b}_2 = \vec{b}_2 - \frac{\vec{b}_2 \cdot \vec{b}_1}{\|\vec{b}_1\|^2} \vec{b}_1$. Here $\vec{b}_2 \cdot \vec{b}_1 = 50$ and $\|\vec{b}_1\|^2 = 25$, so

$$\vec{b}_2' = (2, 11, 12)^T - 2(3, 4, 0)^T = (-4, 3, 12)^T.$$

Normalize: $\|\vec{b}_2'\| = \sqrt{(-4)^2 + 3^2 + 12^2} = \sqrt{169} = 13$, hence

$$\vec{a}_2 = \frac{1}{13}(-4, 3, 12)^T = (-4/13,\ 3/13,\ 12/13)^T.$$